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Asymptotics of odd solutions for cubic nonlinear Schrödinger equations

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ABSTRACT

We consider the Cauchy problem for a cubic nonlinear Schrödinger equation in the case of an odd initial data from $\mathbf{H}^2 \cap \mathbf{H}^{0,2}$. We prove the global existence in time of solutions to the Cauchy problem and construct the modified asymptotics for large values of time.

1. Introduction

We consider the Cauchy problem for the cubic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \sum_{k=-2}^{1} \lambda_k u^{2+k}\bar{u}^{1-k}, & (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\lambda_0 \in \mathbf{R}$ and $\lambda_{-2}, \lambda_{-1}, \lambda_1 \in \mathbf{C}$ in the case of an odd initial data $u_0(x)$.

The difficulty in the study of the global existence in time of solutions to the Cauchy problem (1.1) is that the cubic nonlinear term of Eq. (1.1) is critical for large time values, and it is already known that the usual scattering states do not exist for nonlinear Schrödinger equation (1.1), when the coefficient $\lambda_0 \neq 0$, $\lambda_{-2} = \lambda_{-1} = \lambda_1 = 0$, see [2]. The final value problem of (1.1) was studied extensively in papers [16] and references cited therein. The Cauchy problem for different types of the cubic nonlinearities, including derivatives of the unknown function and the gauge invariant term $|u|^2u$, were considered in papers [6,19,23,24,29] and references cited therein. In paper [10] we considered the nongauge invariant cubic derivative nonlinear Schrödinger equation, when the nonlinearity can be

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represented in the form of the full derivative. In [10] we used the techniques developed in our previous work [8], where we introduced an appropriate representation of the solution and instead of the operator $\mathcal{J} = x + it\partial_x$ we used the dilation operator $x + 2t\partial_t\partial_x^{-1}$ with $\partial_x^{-1} = \int_{-\infty}^x dx$. The nonlinear Schrödinger equation with cubic nonlinearities, containing at least one derivative was studied in paper [22], where the large time asymptotics of solutions was found for small initial data $u_0 \in \mathbf{H}^{3,4}$. The special nonlinearity uu_x^2 or $\bar{u}\overline{u_x}^2$ was considered in [28] and the global existence of small solutions was shown by a different method from [22] and [11] under different conditions on the data from [11]. In paper [11] we improved the previous result of paper [22] by using a different method and estimates in a natural function space $\mathbf{H}^{3,0} \cap \mathbf{H}^{2,1}$. The approach we used in [11] is much more simpler and general than the previous ones in the case of nonlinear terms of the derivative types.

In paper [12] we showed the global existence and the large time asymptotic behavior of solutions to the nonlinear Schrödinger equation (1.1) with nonlinear term $\bar{u}^3$ if the initial data $u(1,x) = u_1 \in \mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}$ are small, and their Fourier transforms are such that $\sup_{|\xi|\leqslant 1}|\arg \widehat{u_1}(\xi)| \leqslant C\varepsilon$, $\inf_{|\xi|\leqslant 1}|\widehat{u_1}(\xi)| \geqslant C\varepsilon$, where and in what follows $\varepsilon > 0$ is a small constant depending on the size of the initial function in a suitable norm. We applied in [12] a method similar to the normal forms of Shatah [25], making a transformation of the cubic nonlinearity $\bar{u}^3$ to new trilinear operators with symbols $(1 + it(\xi^2 + b\eta^2 + c\zeta^2))^{-1}$, with $b, c > 0$. The fact that b and c are positive implies that these trilinear operators have a better time-decay property. We thus obtained the large time asymptotics of solutions which has an additional logarithmic time-decay in the short range region $|x| \leqslant \sqrt{t}$ and quasi-linear in the far region $|x| \gg \sqrt{t}$.

The nonlinear Schrödinger equation (1.1) with the cubic nonlinearity $\lambda_1 u^3 + \lambda_{-1}|u|^2\bar{u} + \lambda_{-2}\bar{u}^3$, $\lambda_1, \lambda_{-1}, \lambda_{-2} \in \mathbf{C}$, was studied in paper [13]. We assumed that there exists $\theta_0 > 0$ such that $\operatorname{Re} \Lambda(r) \geqslant C > 0$ and $r \operatorname{Im} \Lambda(r) \geqslant Cr^2$ for all $|r| < \theta_0$, where the function $\Lambda(r) \equiv \lambda_1 e^{2ir} - i\lambda_{-1}\sqrt{3}e^{-2ir} + \lambda_{-2}e^{-4ir}$. Also we supposed in [13] that the initial data $u_1(x) = u(1,t)$ are such that $\sup_{|\xi|\leqslant 1}|\arg \widehat{u_1}(\xi)| \leqslant C\varepsilon$, $\inf_{|\xi|\leqslant 1}|\widehat{u_1}(\xi)| \geqslant C\varepsilon$. Under these conditions we showed the asymptotics of small solutions. The method was based on the following identity

$$\partial_\xi = i\Upsilon(1 - i\Upsilon\xi)^{-1}(-\widehat{\mathcal{I}} + 2t\partial_t) + (1 - i\Upsilon\xi)^{-1}\partial_\xi$$

with $\widehat{\mathcal{I}} = -\xi\partial_\xi + 2t\partial_t$. Here the parameter $\Upsilon > 0$ has a time growth of order $\sqrt{t}$, so the terms $(1 - i\Upsilon\xi)^{-1}$ appearing in this identity help us to obtain better time-decay properties of the solution. Unfortunately the application of this identity implies some derivative loss with respect to the operator $\widehat{\mathcal{I}}$. Hence to prove the global existence of solutions we need to assume that the initial data belong to some analytic function space.

In paper [14] we proved that if the initial data $u_1 = u(1,t) \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ with a sufficiently small norm $\|u_1\|_{\mathbf{H}^2} + \|u_1\|_{\mathbf{H}^{0,2}} = \varepsilon$ are such that

$$\sup_{|\xi|\leqslant 1}\left|\arg \widehat{u_1}(\xi)\right| < \pi \min\left\{1, \frac{1}{8\omega}\right\}, \qquad \inf_{|\xi|\leqslant 1}\left|\widehat{u_1}(\xi)\right| \geqslant \delta,$$

where $\omega = 1 - \alpha > 0$ and $\delta = \varepsilon^{\frac{6}{5}}$, then there exists a unique solution $u \in \mathbf{C}([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2})$ of the Cauchy problem for the nonlinear Schrödinger equation and the asymptotics of solutions with cubic nonlinearity $\bar{u}^\alpha u^{3-\alpha}$ with $\alpha \in [0,1)$.

In the present paper we consider all the types of cubic nonlinearities in Schrödinger equation, combining gauge invariant term $\lambda_0|u|^2u$ with nongauge invariant terms. If we restrict our attention to odd initial data u_0 then the solution is an odd function for all $t > 0$. This implies that the Fourier transform $\widehat{u_0}$ vanishes at the origin, and as a consequence the nonlinearity obtains better time-decay properties similar to the case of the nonlinear Schrödinger equation with cubic nonlinearities, containing at least one derivative. Note that the condition $\inf_{|\xi|\leqslant 1}|\widehat{u_1}(\xi)| \geqslant \delta$ which was assumed in papers [13,14] cannot be fulfilled for the odd solutions under the consideration in the present paper since we have $\widehat{u_1}(0) = 0$.

Before stating the main result we give some notations. We use the following factorization formulas for the free Schrödinger evolution group $\mathcal{U}(t)\mathcal{F}^{-1} = M(t)\mathcal{D}_t\mathcal{V}(t)$, and $\mathcal{F}\mathcal{U}(-t) = i\mathcal{V}(-t)E(-t)\mathcal{D}_{\frac{1}{t}}$.

Here we denote $M(t) = e^{\frac{i}{2t}x^2}$, $E(t) = e^{\frac{it}{2}\xi^2}$, a dilation operator $(\mathcal{D}_a\phi)(x) = \frac{1}{\sqrt{ia}}\phi(\frac{x}{a})$ and $\mathcal{V}(t) = \mathcal{F}M(t)\mathcal{F}^{-1}$. Note that $\mathcal{D}_{\frac{1}{t}}M(t) = E(t)\mathcal{D}_{\frac{1}{t}}$. The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \hat{\phi} = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-ix\xi}\phi(x)\,dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ix\xi}\phi(\xi)\,d\xi.$$

Denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}';\ \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbf{R}} |\phi(x)|^p\,dx)^{1/p}$ if $1 \leqslant p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. Weighted Sobolev space is

$$\mathbf{H}^{m,k} = \left\{\phi \in \mathbf{S}':\ \|\phi\|_{\mathbf{H}^{m,k}} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\|_{\mathbf{L}^2} < \infty\right\},$$

where $m, k \in \mathbf{R}$, $\langle x\rangle = \sqrt{1+x^2}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}^{m,0}$, so the index 0 we usually omit if it does not cause a confusion. Different positive constants we denote by the same letter C.

Our aim is to prove the following result.

Theorem 1. *Let the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ be odd functions and the norm $\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{H}^{0,2}}$ be sufficiently small. Then there exists a unique global solution u of the Cauchy problem* (1.1) *such that $u \in \mathbf{C}([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2})$. Moreover there exists a unique modified final state $W_+ \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ such that the following asymptotics for $t \to \infty$ is valid*

$$\begin{aligned}
&\left\|u(t) - \frac{1}{\sqrt{it}}W_+\left(\frac{\cdot}{t}\right)\exp\left(\frac{ix^2}{2t} + i\lambda_0\left|W_+\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)\right\|_{\mathbf{L}^\infty} \\
&\quad + t^{\frac{1}{4}}\left\|u(t) - \frac{1}{\sqrt{it}}W_+\left(\frac{\cdot}{t}\right)\exp\left(\frac{ix^2}{2t} + i\lambda_0\left|W_+\left(\frac{\cdot}{t}\right)\right|^2 \log t\right)\right\|_{\mathbf{L}^2} \\
&\leqslant C\varepsilon t^{-\frac{1}{2}-\nu},
\end{aligned} \tag{1.2}$$

where $\nu \in (0, \frac{1}{4})$.

In [16], for the final data belonging to $\mathbf{H}^2 \cap \mathbf{H}^{0,2} \cap \mathring{\mathbf{H}}^{-\gamma,0}$, the modified wave operator was constructed, where $\mathring{\mathbf{H}}^{-\gamma,0}$ is the homogeneous Sobolev space of negative order and $\gamma > \frac{1}{2}$. It is easy to see that if $\phi \in \mathbf{H}^{0,2}$ is an odd function, then $\phi \in \mathbf{H}^{0,2} \cap \mathring{\mathbf{H}}^{-\gamma,0}$ for $\gamma < 1$. Therefore our restrictions on the initial data are stronger than that for the final data in paper [16]. For the convenience of the reader we now briefly explain the main point of the proof. In the same way as in paper [20] we change the variables $u(t,x) = t^{-\frac{1}{2}}Ev(t,\xi)$, $E = e^{\frac{it}{2}\xi^2}$ and $\xi = \frac{x}{t}$ in Eq. (1.1). As we know the estimates of the norm $\|v_\xi\|_{\mathbf{L}^2} = \|x\mathcal{U}(-t)u\|_{\mathbf{L}^2}$ play the crucial role in obtaining the global existence and large time asymptotics of solutions to nonlinear Schrödinger type equations, see [7–14,17,18,21], for example. In order to find the estimates of the norm $\|v_\xi\|_{\mathbf{L}^2}$ we use the transformation similar to the normal forms of Shatah [25] introducing the operator

$$\mathcal{I}(v) = v_\xi - 2t\xi \sum_{k=-2,\,k\neq 0}^{1} A_k E^{2k}\mathcal{N}^{(k)}(v),$$

where

$$A_k = \left(1 + (1+2k)it\xi^2\right)^{-1}, \qquad \mathcal{N}^{(k)}(v) = \lambda_k v^{2+k}\bar{v}^{1-k} \quad \text{for } k = -2, -1, 0, 1.$$

Our proof also depends on the a priori estimate of $\mathcal{FU}(-t)u$ in $\mathbf{L}^\infty$-norm. In order to get the desired estimate we need to take into account the oscillating properties of the nonlinearity.

The Klein–Gordon equation is considered as the relativistic version of the Schrödinger equation and solutions of the Klein–Gordon equation have the same time-decay property as those of the Schrödinger equation. Therefore one can expect the similar results as ours are obtained in nonlinear Klein–Gordon equations with cubic nonlinearities. Indeed nonlinear Klein–Gordon equations with cubic nonlinearities were studied by many authors, and in particular, sharp time decay of solutions was obtained by [3,5,15,26,27] and phase modifications were introduced in [3,15,27] to show global asymptotics of solutions like ours. For time-decay estimate of solutions to the Klein–Gordon equations in general dimensions which are applicable to nonlinear problems with supercritical nonlinearities and space dimensions less than 3, one can see [1,4]. In the case of the Schrödinger equation in general dimension we also have the time-decay estimate which is useful to the nonlinear problems satisfying the self-conjugate property, see [8] for example. However we do not know there is a good estimate which can be applied to general nonlinearities or not.

The rest of the paper is organized as follows. In Section 2 we give some preliminary estimates of the solutions. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

We rewrite our problem (1.1) in the form

$$iu_t + \frac{1}{2}u_{xx} = \sum_{k=-2}^{1} \mathcal{N}^{(k)}(u), \tag{2.1}$$

where $\mathcal{N}^{(k)}(u) = \lambda_k u^{2+k}\bar{u}^{1-k}$ for $k = -2, -1, 0, 1$, and the coefficients $\lambda_0 \in \mathbf{R}$ and $\lambda_{-1}, \lambda_1, \lambda_2 \in \mathbf{C}$. First we state the local existence result.

Theorem 2. *Assume that the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ and $\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{H}^{0,2}} = \tilde{\varepsilon}_0$ is small. Then there exist a time $T = O(\tilde{\varepsilon}_0^{-\frac{2}{3}}) > 1$ and a unique solution $u \in \mathbf{C}([0,T]; \mathbf{H}^2 \cap \mathbf{H}^{0,2})$ of the Cauchy problem (2.1) satisfying the estimate*

$$\sup_{t\in[0,T]} \left(\|u(t)\|_{\mathbf{H}^2} + \|u(t)\|_{\mathbf{H}^{0,2}}\right) \leqslant \tilde{\varepsilon}_0^{\frac{1}{3}}.$$

Proof. Let us consider the linearized problem (2.1) such that

$$iu_t + \frac{1}{2}u_{xx} = \sum_{k=-2}^{1} \mathcal{N}^{(k)}(v),$$

where $\|v\|_{\mathbf{H}^2} + \|v\|_{\mathbf{H}^{0,2}} \leqslant \varepsilon_0 = \tilde{\varepsilon}_0^{\frac{1}{3}}$. Then the usual contraction mapping principle implies the result since we have

$$\|u\|_{\mathbf{H}^2} + \|u\|_{\mathbf{H}^{0,2}} \leqslant CT\left(\tilde{\varepsilon}_0 + \varepsilon_0^3\right) \leqslant CT\varepsilon_0^3 \leqslant \varepsilon_0$$

if we take $T = O(\varepsilon_0^{-2}) = O(\tilde{\varepsilon}_0^{-\frac{2}{3}})$. $\square$

From Theorem 2, we may assume that

$$\left\|u(1)\right\|_{\mathbf{H}^2} + \left\|u(1)\right\|_{\mathbf{H}^{0,2}} \leqslant \tilde{\varepsilon}_0^{\frac{1}{3}} = \varepsilon_0$$

and so we consider the problem for $t \geqslant 1$. In the same way as in paper [20] we change the variables $u(t,x) = t^{-\frac{1}{2}} E v(t,\xi)$, $E = e^{\frac{it}{2}\xi^2}$ and $\xi = \frac{x}{t}$ in Eq. (2.1) to get

$$\begin{cases} \mathcal{L}v = \dfrac{1}{t} \displaystyle\sum_{k=-2}^{1} E^{2k} \mathcal{N}^{(k)}(v), & t \geqslant 1,\ \xi \in \mathbf{R}, \\ v(1,\xi) = v_1(\xi), & \xi \in \mathbf{R}, \end{cases} \tag{2.2}$$

since $\mathcal{FMU}(-t)u = v$, where

$$\mathcal{L} = i\partial_t + \frac{1}{2t^2}\partial_\xi^2.$$

Let the initial datum $v_1(\xi)$ is an odd function, then v is an odd for all $t > 0$. Define the norms

$$\|v\|_{\mathbf{Z}_T} = \sup_{t\in[1,T]} \left\|v(t)\right\|_{\mathbf{L}^\infty}$$

and

$$\|v\|_{\mathbf{Y}_T} = \sup_{t\in[1,T]} \left(t^{-\gamma} \left\|\mathcal{I}(v)\right\|_{\mathbf{L}^2} + t^{-\frac{1}{2}-\gamma} \left\|\partial_\xi \mathcal{I}(v)\right\|_{\mathbf{L}^2}\right),$$

where

$$\mathcal{I}(v) = v_\xi - 2t\xi \sum_{k=-2,\,k\neq 0}^{1} A_k E^{2k} \mathcal{N}^{(k)}(v).$$

Denote $B \equiv \langle\sqrt{t}\xi\rangle^{-1}$. First we prove some auxiliary estimates.

Lemma 1. *Let $v \in \mathbf{H}^2$ be an odd function such that*

$$\|v\|_{\mathbf{Y}_T} + \|v\|_{\mathbf{Z}_T} \leqslant C\varepsilon,$$

with some $\gamma > 0$ and small $\varepsilon > 0$. Then the estimates are true

$$t^{\frac{1}{2}-\gamma}\left\|B^2 v\right\|_{\mathbf{L}^2} + t^{\frac{1}{2}-\frac{3}{2}\gamma}\|Bv\|_{\mathbf{L}^2} + t^{\frac{1}{4}-\gamma}\|Bv\|_{\mathbf{L}^\infty} \leqslant C\varepsilon,$$

$$t^{-\frac{3}{2}\gamma}\|v_\xi\|_{\mathbf{L}^2} + t^{-\gamma}\left\|B^\gamma v_\xi\right\|_{\mathbf{L}^2} + t^{-\gamma-\frac{1}{4}}\|v_\xi\|_{\mathbf{L}^\infty} \leqslant C\varepsilon$$

and

$$t^{-\frac{1}{2}-\frac{3}{2}\gamma}\|Bv_{\xi\xi}\|_{\mathbf{L}^2} + t^{-\frac{1}{2}-\gamma}\left\|B^2 v_{\xi\xi}\right\|_{\mathbf{L}^2} \leqslant C\varepsilon$$

for all $t \in [1,T]$.

Proof. Denote

$$y = v + \sum_{k=-2,\, k\neq 0}^{1} \frac{i}{k} E^{2k} A_k \mathcal{N}^{(k)}(v).$$

Note that

$$y_\xi = \mathcal{I}(v) + \sum_{k=-2,\, k\neq 0}^{1} \frac{i}{k} E^{2k} \partial_\xi \big(A_k \mathcal{N}^{(k)}\big).$$

Since $|A_k| \leqslant CB^2$ and $\sqrt{t}|\xi|B \leqslant C$, we have the estimate

$$|y_\xi| \leqslant \big|\mathcal{I}(v)\big| + C\varepsilon^2\sqrt{t}B^3|v| + C\varepsilon^2 B^2 |v_\xi|.$$

By the definition of $\mathcal{I}(v)$

$$|v_\xi| \leqslant \big|\mathcal{I}(v)\big| + C\varepsilon^2\sqrt{t}B|v|. \tag{2.3}$$

Therefore we have

$$\begin{aligned} |y_\xi| &\leqslant \big|\mathcal{I}(v)\big| + C\varepsilon^2\sqrt{t}B^3|v| + C\varepsilon^2B^2\big(\big|\mathcal{I}(v)\big| + C\varepsilon^2\sqrt{t}B|v|\big) \\ &\leqslant \big(1 + C\varepsilon^2B^2\big)\big|\mathcal{I}(v)\big| + C\varepsilon^2\sqrt{t}B^3|v|. \end{aligned} \tag{2.4}$$

Since y is an odd function in ξ we can write the estimate

$$\big|y(\xi)\big| = \big|y(\xi) - y(0)\big| = \left|\int_0^\xi y_\xi(s)\,ds\right| \leqslant C\|y_\xi\|_{\mathbf{L}^p}|\xi|^{1-\frac{1}{p}}$$

from which with (2.4) we get

$$\begin{aligned} |y| &\leqslant C\sqrt{|\xi|}\|y_\xi\|_{\mathbf{L}^2} \leqslant C\sqrt{|\xi|}\big(\big\|\mathcal{I}(v)\big\|_{\mathbf{L}^2} + \varepsilon^2\sqrt{t}\big\|B^3v\big\|_{\mathbf{L}^2}\big) \\ &\leqslant C\varepsilon\sqrt{|\xi|}\big(t^\gamma + \varepsilon\sqrt{t}\big\|B^3v\big\|_{\mathbf{L}^2}\big). \end{aligned}$$

Then applying the inequality $|v| \leqslant |y| + C\varepsilon^2B^2|v|$ and using the above estimate we obtain for $\delta > 1$

$$\begin{aligned} \big\|B^\delta v\big\|_{\mathbf{L}^2} &\leqslant C\varepsilon^2\big\|B^{2+\delta}v\big\|_{\mathbf{L}^2} + C\varepsilon\big\|B^\delta\sqrt{|\xi|}\big\|_{\mathbf{L}^2}\big(t^\gamma + \varepsilon\sqrt{t}\big\|B^3v\big\|_{\mathbf{L}^2}\big) \\ &\leqslant C\varepsilon^2\big\|B^\delta v\big\|_{\mathbf{L}^2} + C\varepsilon\big(t^{\gamma-\frac{1}{2}} + \varepsilon\big\|B^3v\big\|_{\mathbf{L}^2}\big), \end{aligned}$$

since

$$\big\|B^\delta\sqrt{|\xi|}\big\|_{\mathbf{L}^2}^2 = \int |\xi|\langle\sqrt{t}\xi\rangle^{-2\delta}\,d\xi = t^{-1}\int |\eta|\langle\eta\rangle^{-2\delta}\,d\eta \leqslant Ct^{-1}.$$

Thus we get the estimate

$$\big\|B^\delta v\big\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\gamma-\frac{1}{2}}$$

for $\delta > 1$. In the same manner we see from the inequality $|v| \leqslant |y| + C\varepsilon^2 B^2 |v|$ that

$$\|B^\beta v\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^2 \|B^{\beta+2} v\|_{\mathbf{L}^\infty} + C\varepsilon \|B^\beta \sqrt{|\xi|}\|_{\mathbf{L}^\infty} \big(t^\gamma + \varepsilon\sqrt{t}\|B^3 v\|_{\mathbf{L}^2}\big)$$
$$\leqslant C\varepsilon^2 \|B^\beta v\|_{\mathbf{L}^\infty} + C\varepsilon t^{-\frac{1}{4}+\gamma}$$

for $\beta \geqslant \frac{1}{2}$. This implies the estimate

$$\|B^\beta v\|_{\mathbf{L}^\infty} \leqslant C\varepsilon t^{-\frac{1}{4}+\gamma}$$

for $\beta \geqslant \frac{1}{2}$. For any $\alpha \in [0,1]$ we also can write

$$|y| \leqslant C\varepsilon |\xi|^{\frac{\alpha}{2}} t^{\gamma\alpha}$$

since

$$|y| \leqslant C\varepsilon\sqrt{|\xi|}t^\gamma, \qquad |y| \leqslant C\varepsilon,$$

then for any $\beta > \frac{\alpha+1}{2}$ we find from the inequality $|v| \leqslant |y| + C\varepsilon^2 B^2 |v|$

$$\|B^\beta v\|_{\mathbf{L}^2} \leqslant C\varepsilon^2 \|B^{2+\beta} v\|_{\mathbf{L}^2} + C\varepsilon t^{\gamma\alpha} \|B^\beta |\xi|^{\frac{\alpha}{2}}\|_{\mathbf{L}^2}$$
$$\leqslant C\varepsilon^3 t^{\gamma-\frac{1}{2}} + C\varepsilon t^{\gamma\alpha-\frac{\alpha+1}{4}}.$$

We choose $\frac{\alpha+1}{2} = \beta - \gamma$, then

$$\|B^\beta v\|_{\mathbf{L}^2} \leqslant C\varepsilon \max\big\{t^{\gamma-\frac{1}{2}}, t^{\frac{3}{2}\gamma-\frac{\beta}{2}}\big\}$$

for $\frac{1}{2} \leqslant \beta$, $0 < \gamma$. Therefore we have

$$\|Bv\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\frac{3}{2}\gamma-\frac{1}{2}}, \qquad \|B^2 v\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\gamma-\frac{1}{2}}.$$

Next by estimate (2.3) and by the Sobolev imbedding inequality we have

$$\|v_\xi\|_{\mathbf{L}^\infty} \leqslant \|\mathcal{I}(v)\|_{\mathbf{L}^\infty} + C\varepsilon^2\sqrt{t}\|Bv\|_{\mathbf{L}^\infty}$$
$$\leqslant C\|\mathcal{I}(v)\|_{\mathbf{L}^2}^{\frac{1}{2}} \|\partial_\xi \mathcal{I}(v)\|_{\mathbf{L}^2}^{\frac{1}{2}} + C\varepsilon^2\sqrt{t}\|Bv\|_{\mathbf{L}^\infty} \leqslant C\varepsilon t^{\gamma+\frac{1}{4}}$$

and similarly

$$\|B^\gamma v_\xi\|_{\mathbf{L}^2} \leqslant C\|\mathcal{I}(v)\|_{\mathbf{L}^2} + C\varepsilon^2\sqrt{t}\|B^{1+\gamma} v\|_{\mathbf{L}^2} \leqslant C\varepsilon t^\gamma,$$
$$\|v_\xi\|_{\mathbf{L}^2} \leqslant C\|\mathcal{I}(v)\|_{\mathbf{L}^2} + C\varepsilon^2\sqrt{t}\|Bv\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\frac{3}{2}\gamma}.$$

Finally by estimate

$$|v_{\xi\xi}| \leqslant |\partial_\xi \mathcal{I}(v)| + C\varepsilon^2 t|v| + C\sqrt{t}B|v|^2|v_\xi|$$

we obtain for

$$\|Bv_{\xi\xi}\|_{\mathbf{L}^2} \leqslant \left\|\partial_\xi \mathcal{I}(v)\right\|_{\mathbf{L}^2} + C\varepsilon^2 t\|Bv\|_{\mathbf{L}^2} + C\sqrt{t}\|Bv\|^2_{\mathbf{L}^\infty}\|v_\xi\|_{\mathbf{L}^2}$$
$$\leqslant C\varepsilon t^{\frac{1}{2}+\gamma} + C\varepsilon t^{\frac{1}{2}+\frac{3}{2}\gamma} + C\varepsilon^3 t^{\frac{7}{2}\gamma}$$

and

$$\|Bv_{\xi\xi}\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\frac{1}{2}+\gamma} + C\varepsilon^2 t\left\|B^2 v\right\|_{\mathbf{L}^2} + C\sqrt{t}\|Bv\|^2_{\mathbf{L}^\infty}\|v_\xi\|_{\mathbf{L}^2}$$
$$\leqslant C\varepsilon t^{\frac{1}{2}+\gamma} + C\varepsilon t^{\frac{1}{2}+\gamma} + C\varepsilon^3 t^{\frac{7}{2}\gamma}.$$

Lemma 1 is proved. □

We next prove the following result.

Lemma 2. *Let $v_1 \in \mathbf{H}^2$ be an odd function, and $\|v_1\|_{\mathbf{H}^2} \leqslant \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small. Then the solutions $v \in \mathbf{C}([1,T];\mathbf{H}^2)$ of (2.2) satisfy the estimate*

$$\|v\|_{\mathbf{Y}_T} < \varepsilon,$$

where $\varepsilon = \varepsilon_0^{\frac{1}{3}}$.

Proof. By Theorem 2, we can assume that

$$\|v\|_{\mathbf{Y}_T} + \|v\|_{\mathbf{Z}_T} \leqslant C\varepsilon.$$

Note that from this estimate we have Lemma 1. We prove estimate of the lemma by the contradiction. We assume that there exists a maximal time $\widetilde{T} \in (1,T]$ such that

$$\|v\|_{\mathbf{Y}_{\widetilde{T}}} \leqslant \varepsilon. \tag{2.5}$$

Differentiating (2.2) with respect to ξ we get

$$\mathcal{L}v_\xi = \sum_{k=-2}^{1} 2ik\xi E^{2k}\mathcal{N}^{(k)} + \frac{1}{t}\sum_{k=-2}^{1} E^{2k}\left(\mathcal{N}_v^{(k)} v_\xi + \mathcal{N}_{\bar{v}}^{(k)}\overline{v_\xi}\right), \tag{2.6}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2t^2}\partial_\xi^2$. By a direct calculation we have

$$\mathcal{L}\left(E^{2k}\phi_k\right) = E^{2k}\left(\frac{ik}{tA_k}\phi_k + 2ikt^{-1}\xi\,\partial_\xi\phi_k + \mathcal{L}\phi_k\right) \tag{2.7}$$

with $A_k = (1+(1+2k)it\xi^2)^{-1}$. Then we get from (2.6)

$$\begin{aligned}\mathcal{L}\left(v_\xi + \sum_{k=-2,\,k\neq 0}^{1} E^{2k}\phi_k\right) &= \sum_{k=-2,\,k\neq 0}^{1} \frac{ik}{tA_k}E^{2k}\left(2t\xi A_k\mathcal{N}^{(k)} + \phi_k\right)\\ &\quad + \frac{1}{t}\sum_{k=-2}^{1} E^{2k}\left(\mathcal{N}_v^{(k)} v_\xi + \mathcal{N}_{\bar{v}}^{(k)}\overline{v_\xi} + 2i\xi k\partial_\xi\phi_k\right)\\ &\quad + \sum_{k=-2,\,k\neq 0}^{1} E^{2k}\mathcal{L}\phi_k.\end{aligned} \tag{2.8}$$

To eliminate the first summand in the right-hand side of (2.8) we choose $\phi_k = -2t\xi A_k \mathcal{N}^{(k)}$ for $k \neq 0$. By the identities $\mathcal{L}(\psi\chi) = \chi\mathcal{L}\psi + \frac{1}{t^2}\psi_\xi\chi_\xi + \psi\mathcal{L}\chi$, $\mathcal{L}\overline{\psi} = -\overline{\mathcal{L}\psi} + \frac{1}{t^2}\overline{\psi}_{\xi\xi}$ and

$$\mathcal{L}\mathcal{N}^{(k)} = \mathcal{N}_v^{(k)}\mathcal{L}v - \mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{L}v} + \frac{1}{2t^2}\big(\mathcal{N}_{vv}^{(k)}v_\xi^2 + 2\mathcal{N}_{v\overline{v}}^{(k)}|v_\xi|^2 + \mathcal{N}_{\overline{v}\overline{v}}^{(k)}(\overline{v_\xi})^2\big) + \frac{1}{t^2}\mathcal{N}_{\overline{v}}^{(k)}\overline{v_{\xi\xi}},$$

in view of Eq. (2.2) we obtain

$$\begin{aligned}\mathcal{L}\phi_k = {} & -2\mathcal{N}^{(k)}\mathcal{L}(t\xi A_k) - \frac{2}{t}\partial_\xi(\xi A_k)\big(\mathcal{N}_v^{(k)}v_\xi + \mathcal{N}_{\overline{v}}^{(k)}\overline{v_\xi}\big)\\ & - \frac{1}{t}\xi A_k\big(\mathcal{N}_{vv}^{(k)}v_\xi^2 + 2\mathcal{N}_{v\overline{v}}^{(k)}|v_\xi|^2 + \mathcal{N}_{\overline{v}\overline{v}}^{(k)}(\overline{v_\xi})^2\big) - \frac{2}{t}\xi A_k\mathcal{N}_{\overline{v}}^{(k)}\overline{v_{\xi\xi}}\\ & - 2\xi A_k\sum_{l=-2}^{1}\big(E^{2l}\mathcal{N}_v^{(k)}\mathcal{N}^{(l)} - E^{-2l}\mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(l)}}\big).\end{aligned}\tag{2.9}$$

Therefore we get from (2.8) for the new dependent variable $g \equiv \mathcal{I}(v) = v_\xi - 2t\xi\sum_{k=-2,\,k\neq 0}^{1}A_k E^{2k}\mathcal{N}^{(k)}$

$$\begin{aligned}\mathcal{L}g = {} & -\frac{2}{t}\sum_{k=-2,\,k\neq 0}^{1}E^{2k}\xi A_k\mathcal{N}_{\overline{v}}^{(k)}\overline{v_{\xi\xi}}\\ & - \frac{1}{t}\sum_{k=-2,\,k\neq 0}^{1}E^{2k}\xi A_k\big(\mathcal{N}_{vv}^{(k)}v_\xi^2 + 2\mathcal{N}_{v\overline{v}}^{(k)}|v_\xi|^2 + \mathcal{N}_{\overline{v}\overline{v}}^{(k)}(\overline{v_\xi})^2\big)\\ & + \sum_{k=-2}^{1}E^{2k}\big(t^{-1}b_k\big(\mathcal{N}_v^{(k)}v_\xi + \mathcal{N}_{\overline{v}}^{(k)}\overline{v_\xi}\big) + a_k\mathcal{N}^{(k)}\big)\\ & - 2\sum_{k=-2,\,k\neq 0}^{1}\sum_{l=-2}^{1}\xi A_k\big(E^{2k+2l}\mathcal{N}_v^{(k)}\mathcal{N}^{(l)} - E^{2k-2l}\mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(l)}}\big),\end{aligned}$$

where $a_k = -4ik\xi(\xi A_k)_\xi - 2\mathcal{L}(t\xi A_k)$, $b_k = 1 - 4ikt\xi^2 A_k - 2(\xi A_k)_\xi$ for $k = -2, -1, 1$, and $a_0 = 0$, $b_0 = 1$. Note that $b_k = \frac{1-2k}{1+2k} + f_k$, $f_k = \frac{4k}{1+2k}A_k - 2(\xi A_k)_\xi$. Thus we obtain

$$\mathcal{L}g = -\frac{2}{t}\sum_{k=-2,\,k\neq 0}^{1}E^{2k}\xi A_k\mathcal{N}_{\overline{v}}^{(k)}\overline{g_\xi} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,\tag{2.10}$$

where $\mathcal{R}_1 = \sum_{k=-2}^{1}E^{2k}t^{-1}b_k(\mathcal{N}_v^{(k)}g + \mathcal{N}_{\overline{v}}^{(k)}\overline{g})$,

$$\begin{aligned}\mathcal{R}_2 = {} & -\frac{1}{t}\sum_{k=-2,\,k\neq 0}^{1}\xi A_k E^{2k}\big(\mathcal{N}_{vv}^{(k)}v_\xi^2 + 2\mathcal{N}_{v\overline{v}}^{(k)}|v_\xi|^2 + \mathcal{N}_{\overline{v}\overline{v}}^{(k)}(\overline{v_\xi})^2\big)\\ & - 4\sum_{k=-2,\,k\neq 0}^{1}\sum_{l=-2,\,l\neq 0}^{1}\xi^2 A_k\overline{A_l}E^{2k-2l}\mathcal{N}_{\overline{v}}^{(k)}\big(\overline{\mathcal{N}_v^{(l)}}\,\overline{v_\xi} + \overline{\mathcal{N}_{\overline{v}}^{(l)}}v_\xi\big)\end{aligned}$$

and

$$\mathcal{R}_3 = \sum_{k=-2,\,k\neq 0}^{1} E^{2k} a_k \mathcal{N}^{(k)} + 2\sum_{l=-2}^{1}\sum_{k=-2,\,k\neq 0}^{1} \xi A_k E^{2l+2k}\big(b_l \mathcal{N}_v^{(l)}\mathcal{N}^{(k)} - \mathcal{N}_v^{(k)}\mathcal{N}^{(l)}\big)$$
$$+ 2\sum_{l=-2}^{1}\sum_{k=-2,\,k\neq 0}^{1}\big(\xi \overline{b_l} A_k E^{2k-2l}\mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(l)}} + \xi b_l \overline{A_k} E^{2l-2k}\mathcal{N}_{\overline{v}}^{(l)}\overline{\mathcal{N}^{(k)}}\big).$$

We can write

$$\mathcal{R}_3 = \sum_{k=-4}^{3} E^{2k}\Psi_k,$$

where

$$\Psi_m = \chi^m_{[-2,1]} a_m \mathcal{N}^{(m)} + \chi^m_{[-4,2]} \sum_{k=-2,-1,1,\,m-1\leqslant k\leqslant m+2} 2\xi A_k\big(b_{m-k}\mathcal{N}_v^{(m-k)}\mathcal{N}^{(k)} - \mathcal{N}_v^{(k)}\mathcal{N}^{(m-k)}\big)$$
$$+ 2\xi\chi^m_{[-3,3]}\Bigg(\sum_{k=-2,-1,1,\,m-2\leqslant k\leqslant m+1} \overline{b_{k-m}} A_k \mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(k-m)}}$$
$$+ \sum_{k=-2,-1,1,\,-m-2\leqslant k\leqslant -m+1} b_{k+m}\overline{A_k}\mathcal{N}_{\overline{v}}^{(k+m)}\overline{\mathcal{N}^{(k)}}\Bigg),$$

for $m \neq 0$, where the step-function $\chi^m_{[\alpha,\beta]} = 1$ if $\alpha \leqslant m \leqslant \beta$ and $\chi^m_{[\alpha,\beta]} = 0$ otherwise. Note also that

$$\Psi_0 = 2\xi \sum_{k=-1,1} A_k\big(b_{-k}\mathcal{N}_v^{(-k)}\mathcal{N}^{(k)} - \mathcal{N}_v^{(k)}\mathcal{N}^{(-k)}\big) + 2\xi\sum_{k=-2,\,k\neq 0}^{1}(\overline{b_k}A_k + b_k\overline{A_k})\mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(k)}}$$

which in view of the identity $\mathcal{N}^{(-1)}\mathcal{N}_v^{(1)} = 3\mathcal{N}^{(1)}\mathcal{N}_v^{(-1)}$ can be written as

$$\Psi_0 = -2\xi(8A_1A_{-1} - f_{-1}A_1 - 3f_1A_{-1})\mathcal{N}^{(1)}\mathcal{N}_v^{(-1)}$$
$$+ 2\xi\sum_{k=-2,\,k\neq 0}^{1}\left(\frac{1-2k}{1+2k}2A_k\overline{A_k} + \overline{f_k}A_k + f_k\overline{A_k}\right)\mathcal{N}_{\overline{v}}^{(k)}\overline{\mathcal{N}^{(k)}}. \tag{2.11}$$

Thus we have the equation

$$\mathcal{L}g = -\frac{2}{t}\sum_{k=-2,\,k\neq 0}^{1} E^{2k}\xi A_k \mathcal{N}_{\overline{v}}^{(k)}\overline{g_\xi} + \mathcal{R}_1 + \mathcal{R}_2 + \sum_{k=-4}^{3} E^{2k}\Psi_k. \tag{2.12}$$

To estimate the $\mathbf{L}^2$-norm of g we apply the energy method

$$\frac{d}{dt}\|g\|^2_{\mathbf{L}^2} = -4t^{-1}\operatorname{Re}\sum_{k=-2,\,k\neq 0}^{1}\int_{\mathbf{R}} E^{2k}\xi A_k\mathcal{N}_{\overline{v}}^{k}\overline{g}\,\overline{g_\xi}\,d\xi + 2\operatorname{Re}\int_{\mathbf{R}}\overline{g}\mathcal{R}_1\,d\xi$$
$$+ 2\operatorname{Re}\int_{\mathbf{R}}\overline{g}\mathcal{R}_2\,d\xi + 2\operatorname{Re}\sum_{k=-4}^{3}\int_{\mathbf{R}} E^{2k}\overline{g}\Psi_k\,d\xi. \tag{2.13}$$

Integration by parts yields

$$2\left|\int_{\mathbf{R}} E^{2k}\xi A_k\mathcal{N}_{\bar{v}}^{k}\bar{g}\overline{g_\xi}\,d\xi\right| = \left|\int_{\mathbf{R}} \bar{g}^2\partial_\xi\left(\xi A_k E^{2k}\mathcal{N}_{\bar{v}}^{k}\right)d\xi\right|$$

$$\leqslant C\varepsilon^2\|g\|_{\mathbf{L}^2}^2 + C\varepsilon t^{-\frac{1}{2}}\|g\|_{\mathbf{L}^\infty}\|g\|_{\mathbf{L}^2}^2. \tag{2.14}$$

Next we estimate

$$\left|\int_{\mathbf{R}} \bar{g}\mathcal{R}_1\,d\xi\right| \leqslant C\varepsilon^2 t^{-1}\|g\|_{\mathbf{L}^2}^2. \tag{2.15}$$

Using the estimate

$$|v_\xi| \leqslant |g| + C\varepsilon^2\sqrt{t}B|v|$$

by (2.5) and estimates of Lemma 1 we obtain

$$\left|\int_{\mathbf{R}} \bar{g}\mathcal{R}_2\,d\xi\right| \leqslant C\varepsilon t^{-\frac{3}{2}}\|g\|_{\mathbf{L}^2}\left\|B|v_\xi|^2\right\|_{\mathbf{L}^2} + C\varepsilon^4 t^{-1}\|g\|_{\mathbf{L}^2}\left\|B^2 v_\xi\right\|_{\mathbf{L}^2}$$

$$\leqslant C\varepsilon^3 t^{-1}\|g\|_{\mathbf{L}^2}^2 + C\varepsilon t^{-\frac{3}{2}}\|g\|_{\mathbf{L}^\infty}\|g\|_{\mathbf{L}^2}^2 + C\varepsilon^3 t^{\gamma-1}\|g\|_{\mathbf{L}^2}. \tag{2.16}$$

Now we consider the last term. By (2.11) for the case $m=0$ we find

$$\left|\int_{\mathbf{R}} \bar{g}\Psi_0\,d\xi\right| \leqslant C\varepsilon^4 t^{-\frac{1}{2}}\|g\|_{\mathbf{L}^2}\left\|B^3 v\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^5 t^{\gamma-1}\|g\|_{\mathbf{L}^2}.$$

And finally for the case $m \neq 0$ integrating by parts via the identity

$$E^{2m} = A_{m-\frac{1}{2}}\partial_\xi\left(\xi E^{2m}\right),$$

where $A_{m-\frac{1}{2}} = (1+2imt\xi^2)^{-1}$, we get via estimates of Lemma 1

$$\left|\int_{\mathbf{R}} \bar{g}E^{2m}\Psi_m\,d\xi\right| = \left|\int_{\mathbf{R}} E^{2m}\xi\partial_\xi(A_{m-\frac{1}{2}}\bar{g}\Psi_m)\,d\xi\right|$$

$$\leqslant C\varepsilon^2 t^{-1}\|g_\xi\|_{\mathbf{L}^2}\left\|B^2 v\right\|_{\mathbf{L}^2} + C\varepsilon^4 t^{-\frac{1}{2}}\|g\|_{\mathbf{L}^2}\left\|B^3 v\right\|_{\mathbf{L}^2} + C\varepsilon^2 t^{-1}\|g\|_{\mathbf{L}^2}^2$$

$$\leqslant C\varepsilon^2 t^{\gamma-\frac{3}{2}}\|g_\xi\|_{\mathbf{L}^2} + C\varepsilon^2 t^{\gamma-1}\|g\|_{\mathbf{L}^2} + C\varepsilon^2 t^{-1}\|g\|_{\mathbf{L}^2}^2. \tag{2.17}$$

Collecting estimates (2.14)–(2.17) into (2.13) yields

$$\frac{d}{dt}\|g\|_{\mathbf{L}^2}^2 = C\varepsilon^2 t^{-1}\|g\|_{\mathbf{L}^2}^2 + C\varepsilon t^{-\frac{3}{2}}\|g\|_{\mathbf{L}^\infty}\|g\|_{\mathbf{L}^2}^2 + C\varepsilon^2 t^{\gamma-\frac{3}{2}}\|g_\xi\|_{\mathbf{L}^2}.$$

Hence integrating with respect to time yields

$$\|g\|_{\mathbf{L}^2} \leqslant \varepsilon_0 + C\varepsilon^2 t^\gamma$$

with some $\gamma \in (0, \frac{1}{4})$.

We now differentiate (2.12) with respect to ξ

$$\mathcal{L}g_\xi = 2it\xi \sum_{k=-4,\,k\neq 0}^{3} E^{2k} k(\Psi_k + \Phi_k) - \frac{2}{t} \sum_{k=-2,\,k\neq 0}^{1} E^{2k} \xi A_k \mathcal{N}_{\bar{v}}^{(k)} \overline{g_{\xi\xi}} + \mathcal{R}_4, \tag{2.18}$$

where we denote

$$\Phi_k = t^{-1} \chi_{[-2,1]}^{k} b_k \big(\mathcal{N}_v^{(k)} g + \mathcal{N}_{\bar{v}}^{(k)} \bar{g}\big)$$

and

$$\mathcal{R}_4 = -\frac{2}{t} \sum_{k=-2,\,k\neq 0}^{1} \overline{g_\xi} \partial_\xi \big(E^{2k} \xi A_k \mathcal{N}_{\bar{v}}^{(k)}\big) + \sum_{k=-4}^{3} E^{2k} \partial_\xi (\Psi_k + \Phi_k) + \partial_\xi \mathcal{R}_2.$$

To eliminate the first term in the right-hand side of (2.18) we again use identity (2.7) with $\phi_k = -2t^2 \xi A_k (\Psi_k + \Phi_k)$

$$\mathcal{L}\big(E^{2k}\phi_k\big) = E^{2k}\left(\frac{ik}{tA_k}\phi_k + 2ikt^{-1}\xi \partial_\xi \phi_k + \mathcal{L}\phi_k\right).$$

Then for the new function

$$h = g_\xi - 2t^2 \xi \sum_{k=-4,\,k\neq 0}^{3} A_k E^{2k} (\Psi_k + \Phi_k)$$

we obtain form (2.18)

$$\begin{aligned} \mathcal{L}h = {} & -\frac{2}{t} \sum_{k=-2,\,k\neq 0}^{1} E^{2k} \xi A_k \mathcal{N}_{\bar{v}}^{(k)} \overline{g_{\xi\xi}} + \mathcal{R}_4 + \mathcal{R}_5 \\ & - 2 \sum_{k=-4,\,k\neq 0}^{3} E^{2k} \mathcal{L}\big(t^2 \xi A_k (\Psi_k + \Phi_k)\big), \end{aligned} \tag{2.19}$$

where

$$\mathcal{R}_5 = -4it \sum_{k=-4}^{3} E^{2k} k \xi (\xi A_k)_\xi (\Psi_k + \Phi_k) - 4i \sum_{k=-4}^{3} E^{2k} k t \xi^2 A_k \partial_\xi (\Psi_k + \Phi_k).$$

Now we transform the last term in (2.19) by the identity

$$\mathcal{L}\big(t^2 \xi A_k (\Psi_k + \Phi_k)\big) = t^2 \xi A_k \mathcal{L}\Psi_k + t^2 \xi A_k \mathcal{L}\Phi_k + (\xi A_k)_\xi \partial_\xi (\Psi_k + \Phi_k) + (\Psi_k + \Phi_k)\mathcal{L}\big(t^2 \xi A_k\big)$$

denoting $\widetilde{a_k} = -4itk\xi(\xi A_k)_\xi - 2\mathcal{L}(t^2 \xi A_k)$ and $b_k = 1 - 4ikt\xi^2 A_k - 2(\xi A_k)_\xi$ for $k \neq 0$ and $\widetilde{a_0} = 0$, $b_0 = 1$ we obtain

$$\mathcal{L}h = -2t^{-1} \sum_{k=-2,\,k\neq 0}^{1} E^{2k} \xi A_k (1 + b_k) \mathcal{N}_{\bar{v}}^{(k)} \overline{h_\xi} + \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6, \tag{2.20}$$

where

$$\begin{aligned}
\mathcal{R}_6 = &-2\sum_{k=-4}^{3} E^{2k}(\Psi_k+\Phi_k)\mathcal{L}\big(t^2\xi A_k\big) - 2\sum_{k=-4}^{3} E^{2k}(\xi A_k)_\xi\,\partial_\xi(\Psi_k+\Phi_k)\\
&-4t\sum_{k=-2,\,k\neq 0}^{1}\;\sum_{l=-4,\,l\neq 0}^{3} E^{2k}\xi A_k(1+b_k)\mathcal{N}_{\bar v}^{(k)}\big(E^{-2l}\xi\overline{A_l}(\overline{\Psi_l}+\overline{\Phi_l})\big)_\xi\\
&-2\sum_{k=-4,\,k\neq 0}^{3} E^{2k}t^2\xi A_k\mathcal{L}\Psi_k - 2\sum_{k=-2,\,k\neq 0}^{1} E^{2k}t\xi A_k\big(b_k\mathcal{N}_v^{(k)}\mathcal{L}g - b_k\mathcal{N}_{\bar v}^{(k)}\overline{\mathcal{L}g}\big)\\
&-2\sum_{k=-2,\,k\neq 0}^{1} E^{2k}t\xi A_k\big(g\mathcal{L}\big(b_k\mathcal{N}_v^{(k)}\big)+\bar g\mathcal{L}\big(b_k\mathcal{N}_{\bar v}^{(k)}\big)+t^{-2}g_\xi\big(b_k\mathcal{N}_v^{(k)}\big)_\xi+t^{-2}\overline{g_\xi}\big(b_k\mathcal{N}_{\bar v}^{(k)}\big)_\xi\big).
\end{aligned}$$

By Lemma 1 we have $\|v_\xi\|_{\mathbf{L}^\infty}\leqslant C\varepsilon t^{\gamma+\frac14}$, $\|v_\xi\|_{\mathbf{L}^2}\leqslant C\varepsilon t^{\frac32\gamma}$, $\|B^\gamma v_\xi\|_{\mathbf{L}^2}\leqslant C\varepsilon t^{\gamma}$, $\|Bv_{\xi\xi}\|_{\mathbf{L}^2}\leqslant C\varepsilon t^{\frac12+\frac32\gamma}$ and $\|B^2 v_{\xi\xi}\|_{\mathbf{L}^2}\leqslant C\varepsilon t^{\frac12+\gamma}$. Therefore

$$\begin{aligned}
\|\partial_\xi\mathcal{R}_2\|_{\mathbf{L}^2} &\leqslant C\varepsilon t^{-1}\big(\|v_\xi\|_{\mathbf{L}^\infty}\|v_\xi\|_{\mathbf{L}^2}+\varepsilon t^{\frac12}\|Bv_\xi\|_{\mathbf{L}^2}+\varepsilon\big\|B^2v_{\xi\xi}\big\|_{\mathbf{L}^2}\big)\\
&\quad + Ct^{-\frac32}\big(\|v_\xi\|_{\mathbf{L}^\infty}^2\|Bv_\xi\|_{\mathbf{L}^2}+\varepsilon\|v_\xi\|_{\mathbf{L}^\infty}\|Bv_{\xi\xi}\|_{\mathbf{L}^2}\big)\\
&\leqslant C\varepsilon^3 t^{-\frac12+\gamma}.
\end{aligned}$$

Also we have

$$\begin{gathered}
\|B\Phi_k\|_{\mathbf{L}^2}\leqslant C\varepsilon^2 t^{-1}\|g\|_{\mathbf{L}^2}\leqslant C\varepsilon^3 t^{-1+\gamma},\\
\|\partial_\xi\Phi_k\|_{\mathbf{L}^2}\leqslant C\varepsilon^2 t^{-1}\|g_\xi\|_{\mathbf{L}^2}+C\varepsilon^2 t^{-\frac12}\|g\|_{\mathbf{L}^2}+C\varepsilon t^{-1}\|v_\xi\|_{\mathbf{L}^\infty}\|g\|_{\mathbf{L}^2}\leqslant C\varepsilon^3 t^{-\frac12+\gamma},\\
\|B\Psi_k\|_{\mathbf{L}^2}\leqslant C\varepsilon^2 t^{-\frac12}\big\|B^2 v\big\|_{\mathbf{L}^2}\leqslant C\varepsilon^3 t^{-1+\gamma}
\end{gathered}$$

and

$$\|\partial_\xi\Psi_k\|_{\mathbf{L}^2}\leqslant C\varepsilon^2\big\|B^2v\big\|_{\mathbf{L}^2}+C\varepsilon^2 t^{-\frac12}\|Bv_\xi\|_{\mathbf{L}^2}\leqslant C\varepsilon^3 t^{-\frac12+\gamma}.$$

Therefore we have

$$\begin{aligned}
\|\mathcal{R}_4\|_{\mathbf{L}^2} &\leqslant C\varepsilon^2 t^{-1}\|g_\xi\|_{\mathbf{L}^2}+C\varepsilon t^{-\frac32}\|v_\xi\|_{\mathbf{L}^\infty}\|g_\xi\|_{\mathbf{L}^2}+C\sum_{k=-4}^{3}\big(\|\partial_\xi\Phi_k\|_{\mathbf{L}^2}+\|\partial_\xi\Psi_k\|_{\mathbf{L}^2}\big)+\|\partial_\xi\mathcal{R}_2\|_{\mathbf{L}^2}\\
&\leqslant C\varepsilon^3 t^{-\frac12+\gamma},
\end{aligned}$$

$$\|\mathcal{R}_5\|_{\mathbf{L}^2}\leqslant C\sum_{k=-4}^{3}\big(\sqrt{t}\|B\Phi_k\|_{\mathbf{L}^2}+\sqrt{t}\|B\Psi_k\|_{\mathbf{L}^2}+\|\partial_\xi\Phi_k\|_{\mathbf{L}^2}+\|\partial_\xi\Psi_k\|_{\mathbf{L}^2}\big)\leqslant C\varepsilon^3 t^{-\frac12+\gamma}$$

and finally

$$
\begin{aligned}
\|\mathcal{R}_6\|_{\mathbf{L}^2} \leqslant C \sum_{k=-4}^{3} \left(\sqrt{t}\|B\Phi_k\|_{\mathbf{L}^2} + \sqrt{t}\|B\Psi_k\|_{\mathbf{L}^2} + \|\partial_\xi \Phi_k\|_{\mathbf{L}^2} + \|\partial_\xi \Psi_k\|_{\mathbf{L}^2}\right) \\
&+ C\varepsilon^2 \left\|B^2 v\right\|_{\mathbf{L}^2} + C\varepsilon^2 t^{-1} \left\|B^2 v_{\xi\xi}\right\|_{\mathbf{L}^2} + C\varepsilon t^{-1} \|v_\xi\|_{\mathbf{L}^\infty} \|Bv_\xi\|_{\mathbf{L}^2} \\
&+ C\varepsilon^2 t^{-\frac{1}{2}} \|Bv_\xi\|_{\mathbf{L}^2} + C\varepsilon^2 t^{-1} \|g_\xi\|_{\mathbf{L}^2} + C\varepsilon t^{-\frac{3}{2}} \|v_\xi\|_{\mathbf{L}^\infty} \|g_\xi\|_{\mathbf{L}^2} \\
&+ C\varepsilon t^{-\frac{1}{2}} \|Bv\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^2} + C\varepsilon t^{-1} \|v_\xi\|_{\mathbf{L}^\infty} \|g\|_{\mathbf{L}^2} \\
&+ C\varepsilon t^{-\frac{3}{2}} \|v_\xi\|_{\mathbf{L}^\infty}^2 \|g\|_{\mathbf{L}^2} + C\varepsilon t^{-\frac{3}{2}} \|g\|_{\mathbf{L}^\infty} \|Bv_{\xi\xi}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{-\frac{1}{2}+\gamma}.
\end{aligned}
$$

Then to estimate the $\mathbf{L}^2$-norm of h we apply the energy method to (2.20) and integrate by parts with respect to ξ to avoid the derivative loss

$$
\begin{aligned}
\frac{d}{dt}\|h\|_{\mathbf{L}^2}^2 &\leqslant Ct^{-1} \sum_{k=-2,\, k\neq 0}^{1} \left| \int_{\mathbf{R}} E^{2k} \xi A_k (1+b_k) \mathcal{N}_{\bar{v}}^{k} \overline{h_\xi} \bar{h}\, d\xi \right| + C\varepsilon^3 t^{-\frac{1}{2}+\gamma} \|h\|_{\mathbf{L}^2} \\
&\leqslant C\varepsilon t^{-1} \|h\|_{\mathbf{L}^2}^2 + C\varepsilon^4 t^{2\gamma}.
\end{aligned}
$$

Hence integrating in time we obtain

$$
\|h\|_{\mathbf{L}^2} \leqslant \varepsilon_0 + C\varepsilon^2 t^{\frac{1}{2}+\gamma}
$$

for all $t \in [1, \widetilde{T}]$. Therefore

$$
\|g_\xi\|_{\mathbf{L}^2} \leqslant \|h\|_{\mathbf{L}^2} + Ct^{\frac{3}{2}} \sum_{k=-4}^{3} \left(\|B\Phi_k\|_{\mathbf{L}^2} + \|B\Psi_k\|_{\mathbf{L}^2}\right) < \varepsilon_0 + C\varepsilon^2 t^{\frac{1}{2}+\gamma}
$$

for all $t \in [1, \widetilde{T}]$. Thus we have $\|v\|_{\mathbf{Y}_{\widetilde{T}}} < \varepsilon$. This contradiction proves the estimate of the lemma. Lemma 2 is proved. □

We define the evolution operator

$$
\mathcal{V}(t)\phi = \mathcal{F}M(t)\mathcal{F}^{-1}\phi = \frac{\sqrt{it}}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{it}{2}(\xi-\eta)^2} \phi(\eta)\, d\eta,
$$

then $\mathcal{V}(-t)\mathcal{L}v = i\partial_t(\mathcal{V}(-t)v)$. Note that $\|\mathcal{V}(t)\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$ and the estimates are valid $\|\mathcal{V}(t)\phi\|_{\mathbf{L}^\infty} \leqslant C\sqrt{t}\|\phi\|_{\mathbf{L}^1}$,

$$
\left\|\left(\mathcal{V}(t)-1\right)\phi\right\|_{\mathbf{L}^2} = \left\|\left(e^{\frac{i\xi^2}{2t}}-1\right)\hat{\phi}\right\|_{\mathbf{L}^2} \leqslant Ct^{-\frac{1}{2}} \|\phi_\xi\|_{\mathbf{L}^2} \tag{2.21}
$$

and

$$
\begin{aligned}
\left\|\left(\mathcal{V}(t)-1\right)\phi\right\|_{\mathbf{L}^\infty} &\leqslant C\left\|\partial_\xi\left(\mathcal{V}(t)-1\right)\phi\right\|_{\mathbf{L}^2}^{\frac{1}{2}} \left\|\left(\mathcal{V}(t)-1\right)\phi\right\|_{\mathbf{L}^2}^{\frac{1}{2}} \\
&\leqslant Ct^{-\frac{1}{4}} \|\phi_\xi\|_{\mathbf{L}^2}.
\end{aligned} \tag{2.22}
$$

In the next lemma we obtain the asymptotic representation for the nonlinear term.

Lemma 3. *Let $\alpha \in [0,3]$, $\alpha \neq \frac{3}{2}$. Suppose that $v \in \mathbf{H}^2$ satisfies estimates*

$$\|v\|_{\mathbf{Y}_T} + \|v\|_{\mathbf{Z}_T} \leqslant C\varepsilon,$$

with some $\gamma > 0$ and small $\varepsilon > 0$. Then the following representation is true

$$\mathcal{V}(-t)\big(E^{\beta}\bar{v}^{\alpha}v^{3-\alpha}\big) = E^{\frac{\beta}{\varrho}}\sqrt{i}\mathcal{D}_{\varrho}\bar{\varphi}^{\alpha}\varphi^{3-\alpha} + \mathcal{R}_0(t),$$

where $\varphi(t) = \mathcal{V}(-t)v(t)$, $\beta = 2-2\alpha$, $\varrho = 3-2\alpha$ and $\mathcal{R}_0$ satisfies the estimates

$$\|\mathcal{R}_0\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{1}{4}}, \qquad \|\mathcal{R}_0\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{3}{4}}.$$

Proof. It is easy to compute for all $\beta \in \mathbf{R}$, $\beta \neq -1$

$$\begin{aligned}\mathcal{V}(-t)\big(E^{\beta}\phi\big) &= \sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}} e^{\frac{it}{2}(\xi-\eta)^2+\frac{it}{2}\beta\eta^2}\phi(\eta)\,d\eta \\ &= \sqrt{\frac{t}{2\pi i}}E^{\frac{\beta}{\varrho}}\int_{\mathbf{R}} e^{\frac{it\varrho}{2}(\frac{\xi}{\varrho}-\zeta)^2}\phi(\zeta)\,d\zeta = \sqrt{i}E^{\frac{\beta}{\varrho}}\mathcal{D}_{\varrho}\mathcal{V}(\varrho t)\phi,\end{aligned} \tag{2.23}$$

where $\varrho = 1+\beta$ and $\mathcal{D}_{\varrho}\phi(\xi) = \frac{1}{\sqrt{i\varrho}}\phi(\frac{\xi}{\varrho})$. By (2.23) we find

$$\begin{aligned}\mathcal{V}(-t)\big(E^{\beta}\bar{v}^{\alpha}v^{3-\alpha}\big) &= E^{\frac{\beta}{\varrho}}\sqrt{i}\mathcal{D}_{\varrho}\mathcal{V}(\varrho t)\bar{v}^{\alpha}v^{3-\alpha} \\ &= E^{\frac{\beta}{\varrho}}\sqrt{i}\mathcal{D}_{\varrho}\bar{\varphi}^{\alpha}\varphi^{3-\alpha} + \mathcal{R}_0,\end{aligned}$$

with $\beta = 2-2\alpha$, $\varrho = 3-2\alpha$, $\mathcal{R}_0 = \mathcal{R}_7 + \mathcal{R}_8$, where

$$\mathcal{R}_7 = E^{\frac{\beta}{\varrho}}\sqrt{i}\mathcal{D}_{\varrho}\big(\mathcal{V}(\varrho t)-1\big)\bar{v}^{\alpha}v^{3-\alpha}$$

and

$$\mathcal{R}_8 = E^{\frac{\beta}{\varrho}}\sqrt{i}\mathcal{D}_{\varrho}\big(\bar{v}^{\alpha}v^{3-\alpha} - \bar{\varphi}^{\alpha}\varphi^{3-\alpha}\big).$$

By Lemma 1 we have the estimates $\|v_{\xi}\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{\frac{3}{2}\gamma}$. Therefore by (2.22) we find

$$\|\mathcal{R}_7\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{4}}\big\|\partial_{\xi}\big(\bar{v}^{\alpha}v^{3-\alpha}\big)\big\|_{\mathbf{L}^2} \leqslant C\varepsilon^2 t^{-\frac{1}{4}}\|v_{\xi}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{1}{4}}$$

and

$$\begin{aligned}\|\mathcal{R}_8\|_{\mathbf{L}^\infty} &\leqslant C\big\|\bar{v}^{\alpha}v^{3-\alpha} - \bar{\varphi}^{\alpha}\varphi^{3-\alpha}\big\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^2\big\|\big(\mathcal{V}(-t)-1\big)v(t)\big\|_{\mathbf{L}^2} \\ &\leqslant C\varepsilon^2 t^{-\frac{1}{4}}\|v_{\xi}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{1}{4}}\end{aligned}$$

from which we get

$$\|\mathcal{R}_0\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{1}{4}}.$$

Similarly, by (2.21) we have

$$\|\mathcal{R}_0\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{3}{4}}.$$

Therefore we obtain the result of the lemma. Lemma 3 is proved. □

In the next lemma we obtain the estimates of the function $\varphi(t) = \mathcal{F}\mathcal{U}(-t)u(t) = \mathcal{V}(-t)v(t)$ in the norm $\mathbf{Z}_T$.

Lemma 4. *Let $v_1 \in \mathbf{H}^2$ be an odd function, and $\|v_1\|_{\mathbf{H}^2} \leqslant \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small. Then the solutions $v \in \mathbf{C}([1,T];\mathbf{H}^2)$ of (2.2) satisfy the estimate*

$$\|v\|_{\mathbf{Z}_T} < \varepsilon,$$

where $\varepsilon = \varepsilon_0^{\frac{1}{3}}$.

Proof. We prove estimate of the lemma by the contradiction. We assume that there exists a maximal time $\widetilde{T} \in (1,T]$ such that

$$\|v\|_{\mathbf{Z}_{\widetilde{T}}} \leqslant \varepsilon. \tag{2.24}$$

Then in view of (2.24) via Lemma 3 we can write Eq. (2.2) in the form

$$i\varphi_t = t^{-1}\sum_{k=-2}^{1}\lambda_k E^{\omega_k}\sqrt{i}\mathcal{D}_{1+2k}\overline{\varphi}^{1-k}\varphi^{2+k} + \widetilde{\mathcal{R}}_0(t),$$

where $\varphi(t) = \mathcal{V}(-t)v(t)$, $\omega_k = \frac{2k}{1+2k}$ and

$$\|\widetilde{\mathcal{R}}_0\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{5}{4}}, \qquad \|\widetilde{\mathcal{R}}_0\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma-\frac{3}{2}}.$$

Using the oscillating factor $E^{\frac{2k}{1+2k}}$ we write the identity $E^{\omega_k} = A_{\omega_k}\partial_t(tE^{\omega_k})$ with $A_{\omega_k} = (1+i\omega_k t\xi^2)^{-1}$. Then

$$i\partial_t\left(\varphi - \sum_{k=-2,\,k\neq 0}^{1} A_{\omega_k}E^{\omega_k}\lambda_k\sqrt{i}\mathcal{D}_{1+2k}\overline{\varphi}^{1-k}\varphi^{2+k}\right) = t^{-1}\lambda_0|\varphi|^2\varphi + \mathcal{R}_9 + \widetilde{\mathcal{R}}_0,$$

where

$$\mathcal{R}_9 = t^{-1}\sum_{k=-2,\,k\neq 0}^{1} A_{\omega_k}E^{\omega_k}\lambda_k\sqrt{i}\mathcal{D}_{1+2k}\overline{\varphi}^{1-k}\varphi^{k+2} + \sum_{k=-2}^{1}\omega_k\xi^2 A_{\omega_k}^2 E^{\omega_k}\lambda_k\sqrt{i}\mathcal{D}_{1+2k}\overline{\varphi}^{1-k}\varphi^{k+2}.$$

Denote

$$z = \varphi - \sum_{k=-2,\,k\neq 0}^{1} A_{\omega_k}E^{\omega_k}\lambda_k\sqrt{i}\mathcal{D}_{1+2k}\overline{\varphi}^{1-k}\varphi^{2+k}$$

then we have

$$iz_t = t^{-1}\lambda_0|z|^2 z + \mathcal{R}_9 + \mathcal{R}_{10} + \widetilde{\mathcal{R}}_0, \tag{2.25}$$

with

$$\mathcal{R}_{10} = t^{-1}\lambda_0\big(|\varphi|^2\varphi - |z|^2 z\big).$$

By Lemma 1 we find

$$\begin{aligned}
\|\mathcal{R}_9 + \mathcal{R}_{10}\|_{\mathbf{L}^\infty} &\leqslant Ct^{-1}\|\varphi\|_{\mathbf{L}^\infty}^2\|B\varphi\|_{\mathbf{L}^\infty} + Ct^{-1}\big(\|\varphi\|_{\mathbf{L}^\infty}^2 + \|z-\varphi\|_{\mathbf{L}^\infty}^2\big)\|z-\varphi\|_{\mathbf{L}^\infty} \\
&\leqslant Ct^{-1}\big(\|v\|_{\mathbf{L}^\infty}^2 + \|\varphi - v\|_{\mathbf{L}^\infty}^2\big)\big(\|Bv\|_{\mathbf{L}^\infty} + \|\varphi - v\|_{\mathbf{L}^\infty}\big) \\
&\quad + Ct^{-1}\big(\|v\|_{\mathbf{L}^\infty}^2 + \|\varphi - v\|_{\mathbf{L}^\infty}^2 + \|Bv\|_{\mathbf{L}^\infty}^2\big)\big(\|Bv\|_{\mathbf{L}^\infty} + \|\varphi - v\|_{\mathbf{L}^\infty}\big) \\
&\leqslant C\varepsilon^2 t^{-1}\|Bv\|_{\mathbf{L}^\infty} + C\varepsilon^2 t^{-1}\|\varphi - v\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma - \frac{5}{4}}
\end{aligned}$$

since $\|\varphi - v\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{4}}\|v_\xi\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{-\frac{1}{4}+\frac{3}{2}\gamma}$ and $\|Bv\|_{\mathbf{L}^\infty} \leqslant C\varepsilon t^{-\frac{1}{4}+\gamma}$. Similarly, we get

$$\|\mathcal{R}_9 + \mathcal{R}_{10}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma - \frac{3}{2}}.$$

To exclude the first summand in the right-hand side of (2.25) we make a change $z = \psi \exp(i\lambda_0 \int_1^t \frac{d\tau}{\tau}|z|^2)$

$$i\psi_t = \exp\left(-i\lambda_0 \int_1^t \frac{d\tau}{\tau}|z|^2\right)(\mathcal{R}_9 + \mathcal{R}_{10} + \widetilde{\mathcal{R}}_0) \equiv \mathcal{R}_{11}, \tag{2.26}$$

where

$$\|\mathcal{R}_{11}\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma - \frac{5}{4}}, \qquad \|\mathcal{R}_{11}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{\frac{3}{2}\gamma - \frac{3}{2}}, \tag{2.27}$$

then integrating in time we see that $\|\psi(t)\|_{\mathbf{L}^\infty} \leqslant \varepsilon_0 + C\varepsilon^3$ for all $t \geqslant 1$. Therefore from the identity $v = v - \varphi + \varphi - z + z$, we get

$$\begin{aligned}
\|v\|_{\mathbf{L}^\infty} &\leqslant \|\psi\|_{\mathbf{L}^\infty} + \|z - \varphi\|_{\mathbf{L}^\infty} + \big\|\big(1 - \mathcal{V}(-t)\big)v\big\|_{\mathbf{L}^\infty} \\
&\leqslant \varepsilon_0 + C\varepsilon^3 + C\varepsilon t^{\frac{3}{2}\gamma - \frac{1}{4}} < \varepsilon.
\end{aligned}$$

Thus we have $\|v\|_{\mathbf{Z}_{\widetilde{T}}} < \varepsilon$. This contradiction proves the estimate of the lemma. Lemma 4 is proved. □

3. Proof of Theorem 1

By Lemma 2 a priori estimate of $\|v\|_{\mathbf{Y}_T}$ is shown. On the other hand Lemma 4 states the a priori estimate of $\|v\|_{\mathbf{Z}_T}$. It is easy to see that $\sup_{t\in[1,T]}\|v(t)\|_{\mathbf{H}^{0,2}} = \sup_{t\in[1,T]}\|u(t)\|_{\mathbf{H}^2} \leqslant \varepsilon$ since by Lemmas 2 and 4 we have the estimate

$$\big\|u(t)\big\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{2}}\big\|\mathcal{F}M\mathcal{U}(-t)u(t)\big\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{2}}\big(\|v\|_{\mathbf{L}^\infty} + t^{-\frac{1}{4}+\gamma}\|v_\xi\|_{\mathbf{L}^2}\big).$$

Therefore the global existence of solution $u \in \mathbf{C}([1,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2})$ of the Cauchy problem (2.2) satisfying a priori estimate

$$\|v\|_{\mathbf{Y}_\infty} + \|v\|_{\mathbf{Z}_\infty} + \sup_{t\in[1,\infty)}\big\|v(t)\big\|_{\mathbf{H}^{0,2}} \leqslant 3\varepsilon$$

follows by a standard continuation argument from Lemmas 2, 4 and the local existence Theorem 2.

Now we turn to the proof of the asymptotic formula (1.2) for the solution u of the Cauchy problem (1.1). By (2.26) for the new dependent variable $\psi = z\exp(-i\lambda_0 \int_1^t |z(\tau)|^2 \frac{d\tau}{\tau})$, where

$$z = \varphi - \sum_{k=-2,\, k\neq 0}^{1} A_{\omega_k} E^{\omega_k} \lambda_k \sqrt{i}\mathcal{D}_{1+2k} \overline{\varphi}^{1-k} \varphi^{2+k}$$

and $\varphi(t) = \mathcal{V}(-t)v(t) = \mathcal{FU}(-t)u(t)$, we get

$$i\psi_t = \mathcal{R}_{11}, \tag{3.1}$$

where $\mathcal{R}_{11}$ satisfies the estimates (2.27). Integrating (3.1) in time, we obtain

$$\left\|\psi(t) - \psi(s)\right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 \int_s^t \tau^{\frac{3}{2}\gamma - \frac{5}{4}}\, d\tau \leqslant C\varepsilon^3 s^{-\nu},$$

$$\left\|\psi(t) - \psi(s)\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 \int_s^t \tau^{\frac{3}{2}\gamma - \frac{3}{2}}\, d\tau \leqslant C\varepsilon^3 s^{-\frac{1}{4}-\nu},$$

for any $t > s > 0$, where $\nu \in (0, \frac{1}{4})$. Therefore there exists a unique final state $\psi_+ \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ such that

$$\left\|\psi(t) - \psi_+\right\|_{\mathbf{L}^\infty} + t^{\frac{1}{4}} \left\|\psi(t) - \psi_+\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{-\nu} \tag{3.2}$$

for all $t > 0$. Since $\lambda_0 \in \mathbf{R}$ we write the identity

$$\int_1^t \left|z(\tau)\right|^2 \frac{d\tau}{\tau} = \int_1^t \left|\psi(\tau)\right|^2 \frac{d\tau}{\tau} = |\psi_+|^2 \log t + \Phi(t). \tag{3.3}$$

We study the asymptotics in time of the remainder term $\Phi(t)$. We have

$$\Phi(t) - \Phi(s) = \int_s^t \left(\left|\psi(\tau)\right|^2 - \left|\psi(t)\right|^2\right) \frac{d\tau}{\tau} + \left(\left|\psi(t)\right|^2 - |\psi_+|^2\right) \log \frac{t}{s}.$$

By (3.2) we obtain

$$\left\|\Phi(t) - \Phi(s)\right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 s^{-\nu}, \qquad \left\|\Phi(t) - \Phi(s)\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 s^{-\frac{1}{4}-\nu}$$

for any $t > s > 0$, with some $\nu \in (0, \frac{1}{4})$. Hence there exists a unique real-valued function $\Phi_+ \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ such that

$$\left\|\Phi(t) - \Phi_+\right\|_{\mathbf{L}^\infty} + t^{\frac{1}{4}} \left\|\Phi(t) - \Phi_+\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{-\nu} \tag{3.4}$$

for all $t > 0$. Representation (3.3) and estimate (3.4) yield

$$\left\|\exp\left(i\lambda_0 \int_1^t \left|z(\tau)\right|^2 \frac{d\tau}{\tau}\right) - \exp\left(i\lambda_0\left(|\psi_+|^2 \log t + \Phi_+\right)\right)\right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\nu}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\left\| \varphi - \psi \exp\left(i\lambda_0 \int_1^t \left| z(\tau) \right|^2 \frac{d\tau}{\tau} \right) \right\|_{\mathbf{L}^\infty} = \|\varphi - z\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\nu},$$

$$\left\| \psi \exp\left(i\lambda_0 \int_1^t \left| z(\tau) \right|^2 \frac{d\tau}{\tau} \right) - \psi_+ \exp\bigl(i\lambda_0 \bigl(|\psi_+|^2 \log t + \Phi_+ \bigr) \bigr) \right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\nu}.$$

Therefore

$$\bigl\| \varphi - W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr) \bigr\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\nu}$$

with $W_+ = \psi_+ \exp(i\lambda_0 \Phi_+)$. Similarly,

$$\bigl\| \varphi - W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr) \bigr\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{-\frac{1}{4}-\nu}.$$

Using the factorization of $\mathcal{U}(t)$ we have

$$u(t) = M(t)\mathcal{D}(t)v(t) = M(t)\mathcal{D}(t)\varphi(t) + M(t)\mathcal{D}(t)\bigl(\mathcal{V}(t) - 1\bigr)\varphi(t)$$

from which it follows that

$$\begin{aligned}
&\bigl\| u(t) - M(t)\mathcal{D}(t)\bigl(W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr)\bigr) \bigr\|_{\mathbf{L}^\infty} \\
&\quad \leqslant \bigl\| u(t) - M(t)\mathcal{D}(t)\varphi(t) \bigr\|_{\mathbf{L}^\infty} + \bigl\| M(t)\mathcal{D}(t)\bigl(\varphi(t) - \bigl(W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr)\bigr)\bigr) \bigr\|_{\mathbf{L}^\infty} \\
&\quad \leqslant Ct^{-\frac{1}{2}} \bigl\| \bigl(\mathcal{V}(t) - 1 \bigr)\varphi \bigr\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{2}} \bigl\| \varphi - \bigl(W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr)\bigr) \bigr\|_{\mathbf{L}^\infty} \\
&\quad \leqslant C\varepsilon t^{-\frac{1}{2}-\nu}
\end{aligned}$$

and

$$\bigl\| u(t) - M(t)\mathcal{D}(t)\bigl(W_+ \exp\bigl(i\lambda_0 |W_+|^2 \log t \bigr)\bigr) \bigr\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{-\frac{3}{4}-\nu}.$$

This completes the proof of asymptotics (1.2). Theorem 1 is proved.

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